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by

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by
Chona R. Cerillo[1], Jeffry J. Tejada[2], and Welfredo R. Patungan[3]

ABSTRACT

The paper explores further the power-transformation used in control-charting high precision processes. Charts based on different power-transformations and log-transformation are simulated. Maximum likelihood and method of moments are used to estimate the control limits. Results show that MLE charts based on smaller exponents have smaller MSE than MME charts. However, the latter have greater power and less bias than the former. Both MLE and MME charts based on smaller exponents have similar power as the log-transformation-based chart. The operating characteristic curve shows that charts based on smaller exponents are more sensitive in detecting worsening quality but less sensitive in detecting improving quality. Similarly, the average run lengths of charts based on smaller exponents indicate faster detection of deteriorating quality but higher false-alarm rate.

Keywords and phrases: Control chart, Weibull distribution, Maximum likelihood estimation, Method of moments estimation, Acceptance probability, Average run length

I. Introduction

In high-precision processes , such as semiconductor manufacturing, outgoing quality levels of less than 10 parts-per-million (ppm) nonconforming are routinely achieved (Russell, 1996). With this extremely low defect level, the traditional Shewhart charts are not adequate (Montgomery, 1996). Hence, the development of appropriate control charts is important.

The study is a generalization of the power-transformation-based control chart by Nelson (1994), wherein the 3σ sigma control limits are used for the 0.2777th power of X, the number of items inspected until a nonconforming item is found. This particular value was used in order for the transformed variable to be approximately normally distributed. Charts based on different possible parameter values are simulated and assessed in this study. These charts are also compared with the log-transformation-based chart by McCool and Joyner-Motley (1998). The charts' performances are compared in terms of power, MSE, variance, bias, operating characteristic (OC) curve and average run length (ARL).

II. The Proposed Control Charts

Let X be the number of items inspected until a nonconforming item is found and p the fraction nonconforming in a large production lot. Assuming

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random samples, X is geometrically distributed with probability mass function given by $P(X=x) = p(1-p)^{x-1}$. The same distribution applies for subsequent sampling if the count is reinitialized after each nonconforming item is found.

For a high precision process, p is in parts-per-million (ppm). Hence, the discrete random variable X is well approximated by the continuous exponential distribution having the mean $1/p$, variance $1/p^2$, skewness equal to 2 and kurtosis equal to 9 (McCool and Joyner-Motley, 1998). Since the exponential distribution is a special case of the Weibull distribution in which the shape parameter is equal to 1, then X, as defined earlier, is approximately $W(1/p, 1)$.

In this study the parameter values for the Weibull distribution, other than the numerical value used by Nelson will be explored. Hence, the transformation $Y=X^{0.2777}$ will be generalized as follows:

$$W = X^{\lambda}$$

where $\lambda > 0$.

It is well known that if $X \sim W(\eta, \beta)$, then the transformed variable

$$V = cX^{a} \sim W(c\eta^{a}, \beta/a) \qquad a > 0,\ c > 0 .$$

Thus, if $X \sim W(1/p, 1)$ and $\lambda > 0$,

$$W = X^{\lambda} \sim W((1/p)^{\lambda}, 1/\lambda)$$

For known λ, the maximum likelihood and method of moments estimator for p will be derived and then used to determine the MLE and MME 3σ control limits.

A. Maximum Likelihood Estimation

The power transformation $W = X^{\lambda} \sim W(1/p^{\lambda}, 1/\lambda)$ has the probability density function

$$f(w) = \left(\frac{p}{\lambda}\right) w^{\frac{1}{\lambda}-1} \exp\left\{-p w^{1/\lambda}\right\}$$

where $p > 0$, $\lambda > 0$, $w > 0$.

The likelihood is

$$L(w_1, \dots, w_n; p, \lambda) = \left(\frac{p}{\lambda}\right)^{n} \left(\prod_{i=1}^{n} w_i\right)^{\frac{1}{\lambda}-1} \exp\left\{-p \sum_{i=1}^{n} w_i^{1/\lambda}\right\}$$

and the log-likelihood is

$$\ln L = n(\ln p - \ln \lambda) + \left(\tfrac{1}{\lambda} - 1\right) \sum_{i=1}^{n} \ln w_i - p \sum_{i=1}^{n} w_i^{1/\lambda}$$

Setting the first derivative of the log-likelihood to zero,

$$\frac{\partial \ln L}{\partial p} = \frac{n}{p} - \sum_{i=1}^{n} w_i^{1/\lambda} = 0$$

$$p = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} w_i^{1/\lambda}}$$

yields the MLE

Since $W = X^{\lambda}$,

$$\hat{p} = \left[\frac{1}{n} \sum_{i=1}^{n} x_i \right]^{-1} = \frac{1}{\overline{x}}$$

B. Method of Moments Estimation

Similarly, an estimator for *p* is derived. For $W = X^{\lambda} \sim W(1/p^{\lambda}, 1/\lambda)$,

$$\mu'_1 = E(W) = 1/p^{\lambda}\,\Gamma(1+\lambda)$$

The first sample moment is

$$m'_1 = \sum_{i=1}^{n} w_i = \overline{w}$$

$$\frac{1}{p^{\lambda}}\Gamma(1+\lambda) = \overline{w}$$

Since $\mu'_1 = m'_1$, then

Thus,

$$\tilde{p} = \left[\frac{\Gamma(1+\lambda)}{\overline{w}} \right]^{1/\lambda}$$

C. Simulation

Random samples of sizes n = 30, 50, 100 and 150 are simulated from geometric distributions with *p* = 1, 2, 3, 4, 5,10, 20, 50 and 100ppm. The generated values are used to determine estimates of *p* and the control limits for four types of charts, namely, the power-transformation-based chart , the natural-logarithmic-based chart , the MLE-based chart, and the MME-based chart. For the last two charts which are based on $W = X^{\lambda}$, $\lambda =$ 0.001, 0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.25, 0.2777, 0.5, 1.0, 1.5, 2.0, 2.5, and 3 are used.

For the power-transformation-based chart where $Y = X^{0.2777}$, the observed sample mean is used to obtain a point estimate of *p* by the following equation

$$\hat{p} = \left(\frac{0.901}{\overline{Y}}\right)^{3.6}$$

The control limits set in terms of an observed average value of Y are,

$$LCL = \overline{Y} - 3(0.309)\overline{Y}$$

$$UCL = \overline{Y} + 3(0.309)\overline{Y}$$

For the log-transformation-based chart where *Z* = *ln X*, the sample mean is used to obtain a point estimate of *p* by the equation

$$\hat{p} = \exp\left\{-\left(\overline{Z} + \gamma\right)\right\}$$

where γ is Euler's constant and is approximately equal to 0.5772.

The control limits based on an average value of *Z* = *ln(X)* are given by,

$$LCL = \overline{Z} - 3.849$$

$$UCL = \overline{Z} + 3.849$$

Control charts based on $W = X^{\lambda}$ where λ = 0.001, 0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.25, 0.2777, 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 make use of MLE and MME 3σ control limits.

For the likelihood-based chart, *p* is estimated as follows,

$$\hat{p} = \left[\frac{1}{n}\sum_{i=1}^{n} w_i^{1/\lambda}\right]^{-1}$$

The MLE control limits are determined by the equations,

$$\hat{LCL} = \frac{1}{\hat{p}^{\lambda}}\Gamma(1+\lambda) - \frac{3}{\hat{p}^{\lambda}}\left[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda)\right]^{1/2}$$

$$\hat{UCL} = \frac{1}{\hat{p}^{\lambda}}\Gamma(1+\lambda) + \frac{3}{\hat{p}^{\lambda}}\left[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda)\right]^{1/2}$$

For the moments-based chart, *p* is estimated by the equation

$$\tilde{p} = \left[\frac{\Gamma(1+\lambda)}{\overline{w}}\right]^{1/\lambda}$$

The MME control limits are computed by using the equations

$$L\tilde{C}L = \frac{1}{\tilde{p}^{\lambda}}\Gamma(1+\lambda) - \frac{3}{\tilde{p}^{\lambda}}\left[\Gamma(1+2\lambda) - \Gamma^{2}(1+\lambda)\right]^{1/2}$$

$$U\tilde{C}L = \frac{1}{\tilde{p}^{\lambda}}\Gamma(1+\lambda) + \frac{3}{\tilde{p}^{\lambda}}\left[\Gamma(1+2\lambda) - \Gamma^{2}(1+\lambda)\right]^{1/2}$$

D. Power Function

As a general procedure for each of the four types of charts, the estimated control limits are used for five batches of simulated data values. 1000 values are generated for every batch. For the first batch, the values are generated from a geometric distribution with $p = p_0$ (1, 2, 3, 4, 5,10, 20, 50 and 100ppm). For the second, third, fourth and fifth batches , a process with a shift in p is simulated, that is, the generated values come from geometric distributions with $p = p_1 =$ 50ppm, 100ppm, 200ppm, and 500ppm, respectively.

For the power function, the run rule chosen for signaling a possible out of control condition or the presence of a special-cause variation is a single point plotting beyond the control limits (the most common criterion for a special–cause variation). For the first batch where the simulated data follow an in-control p_0, the number of points that plot beyond the control limits gives the observed probability of type I error. For the second, third, fourth, and fifth batches, the number of points that fall outside the control limits for each batch gives the estimated power.

The following tables are two among several results for the power function.

Table 1

P0	**n**	**Simulated Power of Control Limits**									
		Based on the Average Value of Y = X ^ 0.2777					**Based on the Average Value of Z=lnX**				
(ppm)		p=p0	p=50	p=100	p=200	p=500	p=p0	p=50	p=100	p=200	p=500
1	30	0	5	7	9	31	18	476	748	915	998
	50	0	5	5	8	38	14	484	742	926	999
	100	0	4	8	15	30	18	502	720	939	1000
	150	1	4	5	8	36	17	460	732	932	1000
2	30	2	1	2	9	11	10	225	441	693	940
	50	5	2	0	3	9	10	189	336	586	848
	100	0	1	5	5	11	19	287	486	732	970
	150	2	3	5	11	13	14	243	427	639	942
3	30	1	1	2	2	8	11	128	252	440	804
	50	2	2	5	7	9	13	204	381	616	931
	100	0	1	2	3	9	15	224	385	610	914
	150	1	3	1	4	7	14	149	285	506	838
4	30	0	2	3	5	17	26	230	428	670	906
	50	1	2	1	3	5	14	153	283	480	774
	100	0	1	1	4	12	13	145	275	481	829
	150	2	0	2	5	5	9	129	261	464	794
5	30	2	1	0	1	4	10	92	188	339	641

Table 1 (continued)

		Simulated Power of Control Limits									
P0	**n**	**Based on the Average Value of Y = X ^ 0.2777**					**Based on the Average Value of Z=lnX**				
(ppm)		p=p0	p=50	p=100	p=200	p=500	p=p0	p=50	p=100	p=200	p=500
	50	5	0	1	1	4	9	76	181	291	612
	100	2	1	1	3	6	12	100	194	360	677
	150	1	0	2	0	6	11	99	175	318	620
10	30	5	0	2	1	1	19	54	87	180	373
	50	0	0	1	1	4	14	64	127	218	442
	100	2	0	0	1	5	8	54	85	198	430
	150	0	0	0	0	4	6	51	109	197	449
20	30	0	0	0	1	0	11	33	69	117	267
	50	1	0	0	1	0	12	27	49	95	225
	100	1	0	0	2	0	12	43	49	108	232
	150	0	0	0	0	3	11	20	55	126	273
50	30	0	0	0	0	1	13	12	21	46	104
	50	0	3	0	0	0	9	12	16	38	85
	100	1	0	0	0	1	13	12	22	38	116
	150	1	1	0	0	1	17	13	29	49	107
100	30	3	69	1	0	0	5	7	8	11	45
	50	1	28	0	0	0	19	7	13	28	59
	100	3	31	2	0	0	17	6	13	20	49
	150	0	16	0	0	0	8	3	14	22	65

Table 2

N	**Lambda**	**Simulated Power of Estimated Control Limits for W=X^lambda (p0=5ppm)**									
		MLE-Based Chart					**MME-Based Chart**				
		p=p0	p1=50	p1=100	P1=200	P1=500	P=p0	p1=50	p1=100	p1=200	p1=500
30	0.001	8	85	170	305	587	10	92	187	338	637
	0.005	8	81	168	301	579	8	90	183	333	630
	0.01	7	81	166	295	572	8	88	179	323	621
	0.02	6	74	154	285	560	8	86	174	313	603
	0.03	6	73	151	278	547	8	81	168	301	579
	0.04	6	71	146	268	531	6	77	159	286	562
	0.05	6	68	139	252	510	6	73	151	279	547
	0.1	3	46	106	189	404	4	54	115	205	439
	0.25	2	3	3	14	31	2	3	3	14	35
	0.2777	6	1	0	1	4	2	1	0	1	4
	0.5	16	0	0	0	0	13	0	0	0	0
	1	50	0	0	0	0	50	0	0	0	0
	1.5	61	0	0	0	0	64	0	0	0	0
	2	54	0	0	0	0	59	0	0	0	0
	2.5	47	0	0	0	0	47	0	0	0	0
	3	36	0	0	0	0	36	0	0	0	0
50	0.001	9	76	181	288	612	9	76	181	288	612
	0.005	9	75	178	283	606	9	75	180	286	607
	0.01	8	72	173	276	600	8	73	173	278	603
	0.02	8	71	168	266	576	8	72	168	267	578
	0.03	8	68	158	258	547	8	68	161	260	549
	0.04	8	62	154	245	532	8	62	155	245	536
	0.05	6	59	148	237	507	6	60	148	238	507
	0.1	4	48	105	167	397	4	48	105	169	397
	0.25	5	4	8	12	29	5	4	8	12	29
	0.2777	5	0	1	1	5	5	0	1	1	5
	0.5	19	0	0	0	0	20	0	0	0	0
	1	52	0	0	0	0	52	0	0	0	0

Table 2 (continued)

N	Lambda	Simulated Power of Estimated Control Limits for W=X^lambda (p0=5ppm)									
		MLE-Based Chart					MME-Based Chart				
		p=p0	p1=50	p1=100	p1=200	P1=500	P=p0	p1=50	p1=100	p1=200	p1=500
	1.5	55	0	0	0	0	55	0	0	0	0
	2	54	0	0	0	0	50	0	0	0	0
	2.5	46	0	0	0	0	39	0	0	0	0
	3	37	0	0	0	0	32	0	0	0	0
100	0.001	14	115	207	383	710	12	99	194	360	677
	0.005	13	111	204	377	701	12	99	193	356	671
	0.01	13	103	201	372	697	12	95	189	354	666
	0.02	12	100	194	360	677	12	93	182	338	655
	0.03	12	95	187	351	663	11	90	177	326	643
	0.04	11	93	182	335	651	10	87	169	311	632
	0.05	11	88	174	314	636	9	83	159	303	612
	0.1	9	68	125	244	507	9	66	115	230	483
	0.25	1	2	5	17	50	1	1	3	16	48
	0.2777	1	1	1	3	6	2	1	1	3	6
	0.5	4	0	0	0	0	4	0	0	0	0
	1	15	0	0	0	0	15	0	0	0	0
	1.5	18	0	0	0	0	15	0	0	0	0
	2	15	0	0	0	0	15	0	0	0	0
	2.5	15	0	0	0	0	14	0	0	0	0
	3	13	0	0	0	0	11	0	0	0	0
150	0.001	12	107	189	332	648	11	99	175	318	618
	0.005	12	107	188	332	645	11	98	172	316	613
	0.01	12	104	183	325	636	11	95	171	313	608
	0.02	11	98	173	316	614	9	93	165	302	587
	0.03	10	93	167	308	594	7	90	164	293	578
	0.04	8	92	165	295	583	6	87	158	282	568
	0.05	6	87	158	284	572	6	83	156	271	551
	0.1	6	66	125	221	447	6	63	122	215	435
	0.25	1	2	4	19	34	1	2	4	18	33
	0.2777	0	0	2	0	7	1	0	2	0	7
	0.5	7	0	0	0	0	7	0	0	0	0
	1	23	0	0	0	0	23	0	0	0	0
	1.5	32	0	0	0	0	32	0	0	0	0
	2	25	0	0	0	0	32	0	0	0	0
	2.5	22	0	0	0	0	32	0	0	0	0
	3	15	0	0	0	0	25	0	0	0	0

In general, the power of the charts increases as the magnitude of the shift in *p* increases. The power becomes greater with n but in a considerable number of cases, the power is greater at n=100 than at n=150. The chart based on the average value of *Z* exhibits greater power than the chart based on the average value of *Y*. However, the reverse is true in terms of observed probability of type I error. The MLE-based chart showed less power than the MME-based chart with some exceptions, usually at large sample sizes. The observed probability of type I error for the MLE chart is slightly less than that of the MME chart. For both the MLE and MME charts, power increases as λ decreases. Both MLE and MME charts based on smaller λ have comparable power with the log-transformation-based chart. The highest power attained at close ties by log-transformed, MLE, and MME charts were all at n=100, λ=0.001, and *p*=1ppm.

E. MSE, Variance, and Bias

As a rule, the variance and the MSE of the estimated *p* and the control limits decrease as n increases. The variances and MSE's of the maximum-likelihood estimated *p* and control limits are consistently less than their method-of-moments estimated counterparts. Both variance and MSE decrease as λ becomes smaller. For $\lambda < 1$, MLE bias is greater than that of MME, while the reverse is true for $\lambda > 1$. Both are equal for $\lambda = 1$. For both MLE and MME-based control limits, the bias decreases as n increases. Smaller λ values lead to smaller bias; however, for $0.001 < \lambda < 0.05$,there is not much difference in the bias.

F. The Operating Characteristic (OC) Curve

The operating characteristic (OC) curve is a measure of sensitivity of the control chart to detect a shift in the process. *Pa* values, called acceptance probabilities, for different values of p_1 / p_0 are used to obtain the OC curve.

For $W = X^{\lambda} \sim W(1/p^{\lambda}, 1/\lambda)$,

$$\mu_w = 1/p^{\lambda}\,\Gamma(1+\lambda)$$

and

$$\sigma_w = 1/p^{\lambda}\,[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda)]^{1/2}$$

Thus, the 3σ limits are

$$LCL = \mu_w - 3\sigma_w$$

$$= \frac{1}{p^{\lambda}}\Gamma(1+\lambda) - \frac{3}{p^{\lambda}}\left[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda)\right]^{1/2}$$

$$UCL = \mu_w + 3\sigma_w$$

$$= \frac{1}{p^{\lambda}}\Gamma(1+\lambda) + \frac{3}{p^{\lambda}}\left[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda)\right]^{1/2}$$

The probability that W plots " in control" when p changes to p_1 is given by

$$Pa = P\left(LCL < W < UCL \mid p = p_1\right)$$

$$Pa = F_w\left(UCL \mid p_1\right) - F_w\left(LCL \mid p_1\right)$$

Since $F(w) = 1 - \exp(-pw^{1/\lambda})$

then

$$Pa = 1 - \exp\left(-p_1 UCL^{1/\lambda}\right) - 1 + \exp\left(-p_1 LCL^{1/\lambda}\right)$$

$$Pa = \exp\left(-p_1 LCL^{1/\lambda}\right) - \exp\left(-p_1 UCL^{1/\lambda}\right)$$

$$Pa = \exp\left(-p_1 \left\{ \frac{1}{p_0{}^{\lambda}} \Gamma(1+\lambda) - \frac{3}{p_0{}^{\lambda}} \left[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda) \right]^{1/2} \right\}^{1/\lambda} \right)$$

$$- \exp\left(-p_1 \left\{ \frac{1}{p_0{}^{\lambda}} \Gamma(1+\lambda) + \frac{3}{p_0{}^{\lambda}} \left[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda) \right]^{1/2} \right\}^{1/\lambda} \right)$$

$$Pa = \exp\left(-\frac{p_1}{p_0} \left\{ \Gamma(1+\lambda) - 3\left[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda) \right]^{1/2} \right\}^{1/\lambda} \right)$$

$$- \exp\left(-\frac{p_1}{p_0} \left\{ \Gamma(1+\lambda) + 3\left[\Gamma(1+2\lambda) - \Gamma^2(1+\lambda) \right]^{1/2} \right\}^{1/\lambda} \right)$$

The above formula is used to obtain the acceptance probabilities for $W = X^{\lambda}$, as shown in the following table.

Table 3

P1/p0	**Acceptance Probabilities for W=X^lambda**										***Pa* * for Z=lnX**
	Lambda										
	0.001	**0.005**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.1**	**0.25**	**0.2777**	
0.002	0.0508	0.0490	0.0469	0.0432	0.0400	0.0372	0.0348	0.0261	0.0154	0.0144	0.0513
0.005	0.1222	0.1180	0.1132	0.1045	0.0970	0.0904	0.0846	0.0640	0.0381	0.0357	0.1233
0.01	0.2294	0.2221	0.2136	0.1981	0.1846	0.1727	0.1621	0.1239	0.0747	0.0701	0.2315
0.1	0.9251	0.9179	0.9085	0.8892	0.8692	0.8489	0.8287	0.7331	0.5399	0.5166	0.9270
1	0.9881	0.9883	0.9885	0.9890	0.9895	0.9900	0.9905	0.9931	0.9992	0.9992	0.9882
2	0.9764	0.9768	0.9772	0.9782	0.9791	0.9801	0.9811	0.9863	0.9992	0.9999	0.9765
5	0.9421	0.9430	0.9440	0.9463	0.9486	0.9509	0.9534	0.9661	0.9979	0.9997	0.9422
10	0.8875	0.8892	0.8912	0.8954	0.8998	0.9043	0.9089	0.9333	0.9958	0.9994	0.8878
50	0.5507	0.5558	0.5622	0.5757	0.5899	0.6047	0.6203	0.7082	0.9790	0.9970	0.5516
100	0.3033	0.3089	0.3161	0.3314	0.3479	0.3657	0.3848	0.5015	0.9584	0.9941	0.3042
200	0.0920	0.0954	0.0999	0.1098	0.1211	0.1337	0.1481	0.2515	0.9186	0.9882	0.0926
500	0.0026	0.0028	0.0032	0.0040	0.0051	0.0065	0.0084	0.0317	0.8087	0.9707	0.0026

* *Pa* for Z=lnX (McCool and Joyner-Motley, 1998)

$1 - Pa$ for $p_1 / p_0 = 1$ is simply α. The table shows that the probability of type I error increases as λ decreases from λ=0.2777 to λ=0.001. For $p_1/p_0 > 1$, decreasing *Pa* values for decreasing λ indicates that the chart based on *W* becomes more sensitive in detecting worsening quality as λ becomes smaller. For $p_1/p_0 < 1$, for a ten-fold improvement in quality, the chart becomes less sensitive with smaller λ. The chart based on *W* where λ=0.001 has similar acceptance probability values as the log-transformation-based chart.

G. The Average Run Length (ARL)

The average run length (ARL) is the average number of points that must be plotted before a point indicates an out-of-control condition (Montgomery, 2001). It is a measure of effectiveness of a control chart in such a way that it answers the question: “ Assuming that a shift of some magnitude takes place, how long will it be before we can expect to detect it? “ If the process observations are uncorrelated, then for any Shewhart control chart, the ARL can be computed as,

$$ARL = \frac{1}{\text{P(any point exceeds the control limits)}}$$

If the process is in control,

$$ARL_0 = \frac{1}{\alpha}$$

where α is the probability of type I error.

If the process is out of control,

$$ARL_1 = \frac{1}{1-\beta}$$

where β is the probability of type II error

In this study, the ARL is the average number of observations of X until an out-of-control signal is detected. It is computed as

$$ARL = \frac{1}{1-Pa}$$

where Pa is the probability that the individual value of the variable plots within the control limits given that there was a shift in the process.

The ARL values for W are given by the following table.

Table 4

P1/p0	**Average Run Length for W=X^lambda**										**ARL * for Z=lnX**
	Lambda										
	0.001	**0.005**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.1**	**0.25**	**0.2777**	
0.002	1.05	1.05	1.05	1.05	1.04	1.04	1.04	1.03	1.02	1.01	1.05
0.005	1.14	1.13	1.13	1.12	1.11	1.10	1.09	1.07	1.04	1.04	1.14
0.01	1.30	1.29	1.27	1.25	1.23	1.21	1.19	1.14	1.08	1.08	1.30
0.1	13.35	12.17	10.93	9.02	7.64	6.62	5.84	3.75	2.17	2.07	13.70
1	84.32	85.62	87.33	91.05	95.22	99.91	105.21	145.38	1177.67	1322.44	84.53
2	42.41	43.06	43.92	45.78	47.86	50.21	52.86	72.96	1177.94	8382.15	42.52
5	17.27	17.53	17.87	18.61	19.45	20.39	21.45	29.49	471.58	3366.86	17.31
10	8.89	9.02	9.19	9.56	9.98	10.45	10.98	15.00	236.04	1683.68	8.91
50	2.23	2.25	2.28	2.36	2.44	2.53	2.63	3.43	47.61	337.14	2.23
100	1.44	1.45	1.46	1.50	1.53	1.58	1.63	2.01	24.06	168.82	1.44
200	1.10	1.11	1.11	1.12	1.14	1.15	1.17	1.34	12.28	84.66	1.10
500	1.00	1.00	1.00	1.00	1.01	1.01	1.01	1.03	5.23	34.17	1.00

*ARL for Z=lnX (McCool and Joyner-Motley, 1998)

The table shows that for $p_1/p_0 < 1$ (better quality), specifically for $p_1/p_0=0.1$ (ten-fold improvement in quality), the ARL increases as λ decreases. This implies slower detection of improving quality for smaller λ. However, for much greater decrease in p, the ARL is basically the same across the different λ values. The in-control ARL values for smaller λ, becomes noticeably too short. A shorter in-control ARL implies a higher false-alarm rate (more frequent signaling of out-of-control when the process is actually in control). There is a considerable drop in the out-of-control ARL value as λ decreases from 0.2777 to 0.05. This means faster detection of worsening quality for smaller λ. On the other hand , from $\lambda=0.05$ to $\lambda=0.001$, there is not much difference in the ARL values. The power-transformation-based chart where $\lambda=0.001$ has similar ARL values as the log-transformation-based chart.

III. SUMMARY

In general, the results show that smaller values of λ (i.e. $\lambda<0.2777$) have the following advantages:

1. greater power of the test
2. smaller MSE, variance, and bias of the estimated p and control limits
3. the control chart becomes more sensitive in detecting an increase in p due to smaller acceptance probability (Pa) for $p_1/p_0> 1$
4. faster detection of worsening quality as shown by shorter out-of-control ARL values

However, smaller values of λ have the following drawbacks:

1. the chart becomes less sensitive in detecting improving quality (as shown by increasing Pa for $p_1/p_0> 1$)
2. in-control ARL value becomes unacceptably short (higher false-alarm rate)

Thus, based on the advantages of smaller λ, the control charts based on $W=X^{\lambda}$ where $\lambda<0.2777$ perform better than the chart based on $Y=X^{0.2777}$.

The power-transformation-based chart where $\lambda=0.001$ has similar acceptance probability and average run length as the log-transformation-based chart.

MLE-based charts have smaller MSE, variance, and observed probability of type I error than MME-based charts. However, MME-based charts show greater power and smaller bias than MLE-based charts. Both MLE and MME charts based on smaller λ have comparable power with the log-transformation-based chart.

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