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Sequential Point Estimation of Functions of the Exponential Scale Parameter

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ABSTRACT

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables according to an exponential distribution with scale parameter $\sigma \in (0, \infty)$, assumed to be unknown. We propose a purely sequential method for the estimation of positive, three-times continuously differentiable functions of the unknown exponential scale, $\theta = \theta(\sigma)$. Given a sample $X_1, X_2, \ldots, X_n$ of size n, we wish to estimate the function $\theta = \theta(\sigma)$ by $\hat{\theta}_n = \theta(\bar{X}_n)$, subject to the loss function $L(\hat{\theta}_n) = (\hat{\theta}_n - \theta) + cn$, where c is the cost per unit sample. We estimate the optimum sample size n^* that minimizes the risk $R_n = E\{L(\hat{\theta}_n)\}$, sequentially. First, we start with a random sample $X_1, \ldots, X_m$ of initial size $m > 1$, taken from the exponential population. If $m < c^{-\frac{1}{2}} \bar{X}_m |\theta'(\bar{X}_m)|$, then we take one more observation from the population. Sampling is continued until we find the sample size, say k, so that $k \geq c^{-\frac{1}{2}} \bar{X}_k |\theta'(\bar{X}_k)|$. Thus, the proposed estimate of n^* is the stopping rule $N = \inf\{n \geq m : n \geq c^{-\frac{1}{2}} \bar{X}_n |\theta'(\bar{X}_n)|\}$, and hence estimate θ by $\hat{\theta}_N = \theta(\bar{X}_N)$. The performance of the procedure is measured by the regret $R_N - 2cn^*$. For a fully sequential sampling scheme, we derive second-order approximations of the expected sample size and the risk of the proposed sequential procedure as $c \to 0$. As an application, we consider the estimation of the hazard rate $\theta = \sigma^{-1}$ with simulation results. On the hazard rate, we show that our sequential procedure attains the minimum risk $2cn^*$ up to the order term.

I. Introduction

Let X_1, X_2, … be independent and identically distributed random variables according to an exponential distribution having the probability density function given by

$$f_s(x) = \frac{1}{s} exp\left(-\frac{x}{s}\right), \quad x > 0,$$

where the scale parameter $\sigma \in (0, \infty)$ is unknown. It is interesting to estimate the mean σ and the variance σ^2. We consider the more general case of estimating functions of the scale parameter σ.

Suppose that $q(x)$ is a positive-valued and three-times continuously differentiable function on $\{x > 0\}$. Let q', q'' and $q^{(3)}$ denote the first, second and third derivatives of q, respectively. Assume that $q'(x) \neq 0$ for all values of $x > 0$. Given a random sample of X_1, X_2, …, X_n of size n, we estimate the function $q = q(s)$ by $\hat{q}_n = q(\overline{X}_n)$, where the sample mean $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$. The accuracy of the estimate is measured by the *loss function*

$$L(\hat{q}_n) = (\hat{q}_n - q)^2 + cn,$$

where $c > 0$ is the known cost per unit sample. The *risk* is given by

$$R_n = E\{L(\hat{q}_n)\} = E(\hat{q}_n - q)^2 + cn.$$

One goal is to find the appropriate sample size that will minimize the risk. Under a certain condition, for sufficiently large n, $R_n \approx s^2\{q'(s)\}^2 n^{-1} + cn$, which is approximately minimized at

$$n_0 \approx \frac{s\left|q'(s)\right|}{\sqrt{c}} = n^*, \text{ say.} \tag{1.1}$$

It follows that $R_{n_0} \approx 2cn^*$. However, since σ is unknown, we cannot use the optimum fixed sample size n^*. Also, according to Takada [2], there is no fixed sample size procedure that will attain the minimum risk R_{n^*}. Thus, it is necessary to find a sequential sampling procedure. Motivated by the optimum sample size n^* in (1.1), we propose the following sequential procedure. First, take a pilot sample of size m = 2, from the exponential population. If $m < c^{-1/2}\overline{X}_m\left|q'(\overline{X}_m)\right|$, then we take one more observation, x_{m+1}, otherwise sampling is terminated. Sampling continues or stops based on the following stopping rule:

$$N = N_c = \inf\left\{n \geq m : n \geq \frac{\overline{X}_n\left|q'(\overline{X}_n)\right|}{\sqrt{c}}\right\}. \tag{1.2}$$

In this case, N is our proposed estimate of the unknown optimum sample size n^*. Once sampling is stopped after taking N observations, we estimate $q = q(s)$ by $\hat{q}_N = q(\overline{X}_N)$ with the risk of the estimate given by $R_N = E(\hat{q}_N - q)^2 + cE(N)$. The performace of the procedure is measured by the regret $R_N - 2cn^*$.

In the succeeding section, we will give second-order approximations of the expected sample size $E(N)$ and asymptotic expansions of the risk R_N of the sequential procedure.

II. Main Results

Let

$$h(x) = \frac{1}{x\sqrt{\{q'(x)\}^2}} \quad \text{for } x > 0.$$

Then the stopping rule in (1.2) becomes

$$N = \inf\{n \geq m : Z_n \geq n^*\} \quad \text{where } \; Z_n = n\frac{h(\overline{X}_n)}{h(s)}.$$

Let $Y_i = (X_i / s) - 1$ for $i = 1, 2, \ldots$, $S_n = \sum_{i=1}^{n} Y_i$ and $\overline{Y}_n = n^{-1}S_n$. By Taylor's Theorem,

$$h(\overline{X}_n) = h(s) + h'(s)(\overline{X}_n - s) + \tfrac{1}{2}(\overline{X}_n - s)^2 h''(h_n),$$

where h_n is a random variable lying between σ and $\overline{X}_n$. Then we can write Z_n as

$$Z_n = n + a S_n + \xi_n$$

where

$$a = -\left(1 + \frac{s q''(s)}{q'(s)}\right) \quad \text{and} \quad x_n = n(\overline{X}_n - s)^2 \frac{h''(h_n)}{2h(s)}, \tag{2.1}$$

with

$$h''(x) = \frac{q'(x)}{|q'(x)|}\left\{2\frac{\{q'(x) + xq''(x)\}^2}{x^3\{q'(x)\}^3} - \frac{2q''(x) + xq^{(3)}(x)}{x^2\{q'(x)\}^2}\right\}.$$

Let

$$t = \inf\{n \geq 1 : n + aS_n > 0\} \quad \text{and} \quad r = \frac{E(t + aS_t)^2}{2E(t + aS_t)}. \tag{2.2}$$

Consider also the following assumptions:

(A1) $\left\{\left[\left(Z_n - \dfrac{n}{e_0}\right)^+\right]^3, n \geq m\right\}$ is uniformly integrable for some $0 < e_0 < 1$,

where $x^+ = \max(x, 0)$.

(A2) $\sum_{n=m}^{\infty} nP\{x_n < -e_1 n\} < \infty$ for some $0 < e_1 < 1$.

Then we obtain the following approximation to the expected sample size for all σ ? (0, ∞) but not uniformly in σ.

Theorem 2.1. *If* (*A*1) *and* (*A*2) *hold, then*

$$E(N) = n^* + r - l + o(1) \quad \text{as } c \to 0,$$

where

$$l = 1 + \frac{sq''(s)}{q'(s)} + \frac{s^2\{q''(s)\}^2}{\{q'(s)\}^2} - \frac{s^2 q^{(3)}(s)}{2q'(s)}.$$

Proof. Let W be distributed according to a standard normal distribution $N(0, 1)$. Now,

$$x_n = s^2 \left[\frac{\overline{X}_n - s}{s / \sqrt{n}} \right]^2 \frac{h''(h_n)}{2h(s)},$$

and

$$x_n \xrightarrow{d} \left[1 + \frac{sq'(s)}{q'(s)} + \frac{s^2\{q'(s)\}^2}{\{q'(s)\}^2} - \frac{s^2 q^{(3)}(s)}{2q'(s)} \right] \cdot W^2.$$

Thus, $x_n \xrightarrow{d} x \equiv l \cdot W^2$ as $n \to \infty$, where "$\xrightarrow{d}$" denotes convergence in distribution. We shall check conditions (C1) to (C6) of Aras and Woodroofe [1]. Clearly, (C1) holds. (C2) with $p = 3$ and (C3) are identical with (A1) and (A2), respectively. Letting $g(y) = h(sy + s)/h(s)$, (C4), (C5) and (C6) follow from Proposition 4 of Aras and Woodroofe [1]. Hence, from Theorem 1 of Aras and Woodroofe [1],

$E(N) = n^* + r - E(x) + o(1) = n^* + r - l + o(1)$ as $c \to 0$,

which concludes the theorem. €

The proposition below gives sufficient conditions for (A2) which are useful in actual estimation problems.

Proposition 2.1 .

(i.) *If* $h''(h_n) \geq 0$ *for all* $n = m$, *then* (*A*2) *holds.*

(ii.) *If* $\sup_{n \geq m} E\left|h''(h_n)\right|^s < \infty$ *for some* $s > 2$, *then* (*A*2) *holds.*

Proof. If $h''(h_n) \geq 0$, then $x_n \geq 0$ for all $n = m$. Clearly (A2) is satisfied. Suppose $\sup_{n \geq m} E\left|h''(h_n)\right|^s < \infty$ *for some* $s > 2$, and let $0 < \varepsilon < 1$ and $q > 2$. Then, by the Markov and Holder's Inequalities, we obtain

$$\begin{aligned} P(x_n < -en) &= P\{-h''(h_n)(\overline{X}_n - s)^2 > 2h(s)e\} \\ &\leq (2h(s)e)^{-q} E\left|(\overline{X}_n - s)^2 h''(h_n)\right|^q \\ &\leq (2h(s)e)^{-q} \left\{E\left|(\overline{X}_n - s)^{2qu}\right|\right\}^{\frac{1}{u}} \left\{E\left|h''(h_n)\right|^{qv}\right\}^{\frac{1}{v}}, \end{aligned}$$

where $u^{-1}+v^{-1}=1$ and $u>1$. Choose (u, v) and q > 2 such that s = qv. By the Marcinkiewicz-Zygmund Inequality, we have $E\left|(\overline{X}_n - s)\right|^{2qu} = O(n^{-qu})$. Thus, we have $nP(x_n < -en) = O(n^{1-q})$ as $n \to \infty$, which implies (A2).

The performace of the proposed sequential procedure is measured by the regret given as the difference between the risk of the estimate computed from a random sample of size N and the approximately minimum risk $2cn^*$. Let us now assess the regret R_N - $2cn^*$. By Taylor's Theorem,

$$q(\overline{X}_N) - q(s) = q'(s)(\overline{X}_N - s) + \tfrac{1}{2}q''(s)(\overline{X}_N - s)^2 + \tfrac{1}{6}(\overline{X}_N - s)^3 q^{(3)}(f_c),$$

where f_c is a random variable lying between σ and $\overline{X}_N$. Choose c_0 > 0 such that n^* = 1. We impose the following assumption:

(A3) For some a > 1 and u > 1,

$$\sup_{0<c\le c_0} E\left|(n^*)^{\frac{1}{2}}\overline{Y}_N\right|^{4au} < \infty \quad \text{and} \quad \sup_{0<c\le c_0} E\left|q^{(3)}(f_c)\right|^{2au/(u-1)} < \infty..$$

Uno, Isogai and Lim-Polestico [4] evaluated the risk as given in the next theorem. The reader is referred to [4] for a detailed proof of this important result.

Theorem 2.2. *If* (*A*1), (*A*2) *and* (*A*3) *hold, then*

$$R_N - 2cn^* = \left\{3 + 2\frac{sq''(s)}{q'(s)} + \frac{7s^2\{q''(s)\}^2}{4q'\{(s)\}^2} - \frac{s^2 q^{(3)}(s)}{q'(s)}\right\}c + 0(c) \quad \text{as } c \to 0.$$

Remark 2.1. *Theorem 2.2 shows that in estimating the mean* $q = s$, *the regret becomes* $3c + o(c)$, *which coincides with the result of Woodroofe* [5].

III. Example: The Exponential Hazard Rate

As an example, let us consider the estimation of the hazard rate $q = s^{-1}$.
In estimating q by $\hat{q}_n = \overline{X}_n^{-1}$, the risk is given by

$$R_n = E\{L(\hat{q}_n)\} = E\{(\hat{q}_n - q)^2 + cn\} = E(\overline{X}_n^{-1} - s^{-1})^2 + cn,$$

which is finite for n > 2. We can also show that

$$R_n = \frac{n+2}{(n-1)(n-2)}s^{-2} + cn = s^{-2}n^{-1} + cn + O(n^{-2}) \quad \text{as } n \to \infty.$$

From (1.1) the optimum sample size $n^* = c^{-1/2}s^{-1}$ and the stopping rule N in (1.2) becomes

$$N = \inf\left\{n \geq m : n \geq c^{-\frac{1}{2}} \overline{X}_n^{-1}\right\}.$$

A second-order approximation to the expected sample size is given in the next theorem.

Theorem 3.1. *Suppose* $m = 1$. Then

$$E(N) = n^* + 1 + o(1) \quad \text{as} \quad c \to 0.$$

Proof. Define $g(x) = x + 1$. Then,

$$Z_n = n\overline{X}_n / s = ng(\overline{Y}_n),$$

where $\overline{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i = n^{-1}\sum_{i=1}^{n}(X_i / s - 1)$. Since $g(x)$ is convex and $E[\{g(Y_1)^+\}]^3 = E(X_1/s)^3 < \infty$, from Proposition 5 of Aras and Woodroofe [1], (A1) and (A2) are satisfied. Now, since $q = s^{-1}$, we have

$$a = -\left(1 + \frac{sq''(s)}{q'(s)}\right) = 1 \quad \text{and} \quad l = 1 + \frac{sq''(s)}{q'(s)} + \frac{s^2\{q''(s)\}^2}{\{q'(s)\}^2} - \frac{s^2 q^{(3)}(s)}{2q'(s)} = 0.$$

The stopping time t in (2.2) becomes $t = \inf\{n = 1 : n + S_n > 0\} = 1$, so that

$$r = \frac{E(t + aS_t)^2}{2E(t + aS_t)} = \frac{E(1 + Y_1)^2}{2E(1 + Y_1)} = 1.$$

Hence, Theorem 2.1 with $l = 0$ and $r = 1$ yield the theorem. €

We then estimate $q = s^{-1}$ by $\hat{q}_N = \overline{X}_N^{-1}$, and obtain the risk as

$$R_N = E\{L(\hat{q}_N)\} = E(\overline{X}_N^{-1} - s^{-1})^2 + cE(N).$$

Theorem 3.2 that follows gives us the regret of the sequential procedure.

Theorem 3.2. *If* $m > 12$, *then*

$$R_N - 2cn^* = o(c) \quad as \quad c \to 0.$$

It follows from Theorem 3.2 that the proposed procedure attains the minimum risk $2cn^*$ up to the $o(c)$ term.

IV. Simulation

In order to evaluate the results given in the previous sections, we shall include brief simulation results. We are interested in the performance of our sequential procedure for various values of s, so we consider the cases when s = 0.5, 1 and 2 with corresponding q = 2, 1 and 0.5. Since the cost c is sufficiently small in our theorems, the values of c are chosen such that $n^* = c^{-1/2}s^{-1} = 30$ and the initial sample size is set at m = 13. From Theorem 3.1 and Theorem 3.2, we have

$$E(N) = n^* + 1 + o(1) \quad \text{and} \quad \frac{R_N - 2cn^*}{c} = \mathrm{o}(1) \quad \text{as} \quad c \to 0.$$

The results in Table 4.1 which are based on 100,000 repetitions by means of the stopping rule N, seem to justify our theorems. Also, on the average, the estimate of the unknown hazard rate is very close to the true value. Thus, our sequential procedure seems to be effective and useful.

Table 4.1. $n^* = 30$

m = 13	$s = 0.5$	$s = 1$	$s = 2$
	c = 0.0044 q = 2	c = 0.0011 q = 1	c = 0.000277 q = 0.5
$E(N)$	30.996970	31.031260	30.997910
$E(\hat{q}_N)$	2.001831	1.002166	0.500437
$(R_N - 2cn^*)/c$	- 0.104954	- 0.201335	- 0.222469

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[1] G. Aras and M. Woodroofe. Asymptotic expansions for the moments of a randomly stopped average. The *Annals of Statist.,* 21 : 503-519, 1993.

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